Financial Engineering & Risk Management

Introduction to Mortgage Mathematics and Mortgage Backed Securities

M. Haugh **G. Iyengar**

Department of Industrial Engineering and Operations Research
Columbia University

Mortgage-Backed-Securities Markets

Recall that according to *SIFMA*, in Q3 2012 the total outstanding amount of US bonds was $\$35.3$ trillion:

Government	\$10.7	30.4%
Municipal	\$3.7	10.5%
Mortgage	\$8.2	23.3%
Corporate	\$8.6	24.3%
Agency	\$2.4	6.7%
Asset-backed	\$1.7	4.8%
Total	\$35.3 tr	100%

– the mortgage market accounted for 23.3% of this total!

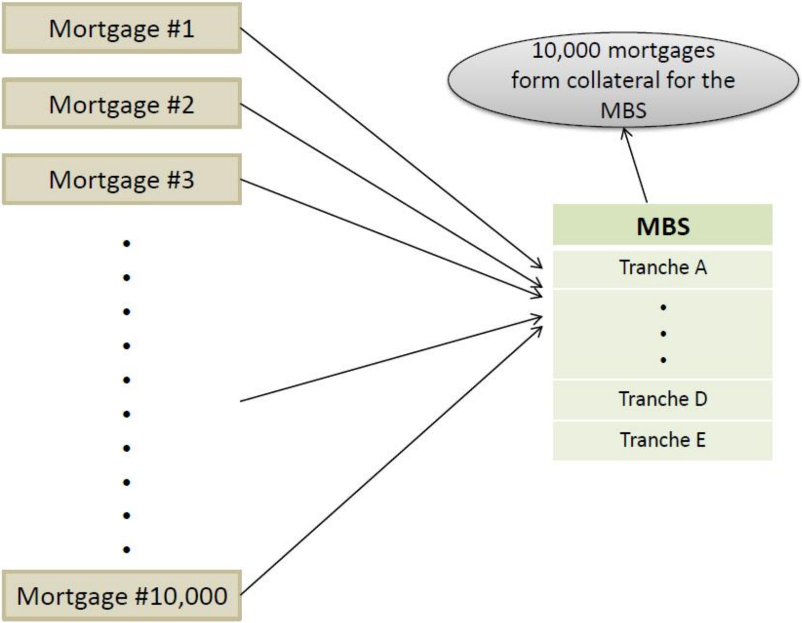
The mortgage markets are therefore huge

– and played a big role in the financial crisis of 2008 / 2009.

MBS are a particular class of asset-backed securities (ABS)

– assets **backed** by underlying pools of securities such as mortgages, auto-loans, credit-card receivables, student loans etc.

– the process by which ABS are created is often called securitization.


Mortgage #1
Mortgage #2
Mortgage #3
Mortgage #10,000
10,000 mortgages form collateral for the MBS
MBS
Tranche A
Tranche D
Tranche E

MBS Markets

We will look at some examples of MBS but first must consider the mathematics of the underlying mortgages.

There are many different types of mortgages including:

1. level-payment mortgages
2. adjustable-rate mortgages (ARMs)
3. balloon mortgages
4. and others.

We will only consider level-payment mortgages

– but MBS may be constructed out of other mortgage types as well.

The construction of MBS is an example of securitization

– the same ideas apply to asset-backed securities more generally.

A standard reference on mortgage-backed securities is *Bond Markets, Analysis and Strategies* (Pearson) by F.J Fabozzi. But it is very expensive!

Basic Mortgage Mathematics for Level-Payment Mortgages

We consider a standard level-payment mortgage:

- Initial mortgage principal is $M_0 := M$.
- We assume equal periodic payments of size B dollars.
- The coupon rate is c per period.
- There are a total of n repayment periods.
- After the n payments, the mortgage principal and interest have all been paid
  - the mortgage is then said to be fully **amortizing**.

This means that each payment, B, pays both interest and some of the principal.

If M_k denotes the mortgage principal remaining after the k^{th} period then

$$M_k = (1+c)M_{k-1} - B \quad \text{for} \quad k = 0, 1, 2, \ldots, n \tag{1}$$

with $M_n = 0$.

Can iterate (1) to obtain

$$\begin{aligned} M_k &= (1+c)^k M_0 - B \sum_{p=0}^{k-1} (1+c)^p \\ &= (1+c)^k M_0 - B \left[\frac{(1+c)^k - 1}{c} \right]. \end{aligned} \tag{2}$$

But $M_n = 0$ and so we obtain

$$B = \frac{c(1+c)^n M_0}{(1+c)^n - 1}. \tag{3}$$

Can now substitute (3) back into (2) and obtain

$$M_k = M_0 \frac{(1+c)^n - (1+c)^k}{(1+c)^n - 1}. \tag{4}$$

The Present Value of a Level-Payment Mortgage

Suppose now that we wish to compute the present value of the mortgage assuming a deterministic world

- with no possibility of defaults or prepayments.

Then assuming a risk-free interest rate of r per period, we obtain that the fair mortgage value as

$$
\begin{aligned}
F_0 &= \sum_{k=1}^{n} \frac{B}{(1+r)^k} \\
&= \frac{c(1+c)^n M_0}{(1+c)^n - 1} \times \frac{(1+r)^n - 1}{r(1+r)^n}.
\end{aligned} \tag{5}
$$

Note that if $r = c$ then (5) immediately implies that $F_0 = M_0$

– as expected!

In general, however, $r < c$, to account for the possibility of default, prepayment, servicing fees, profits, payment uncertainty etc.

Scheduled Principal and Interest Payments

Since we know M_{k-1} we can compute the interest

$$I_k \;:=\; cM_{k-1}$$

on M_{k-1} that would be due in the next period, i.e. period k.

This also means we can interpret the k^{th} payment as paying

$$P_k :=\; B - cM_{k-1}$$

of the remaining principal, M_{k-1}.

So in any time period, k, we can easily break down the payment B into a scheduled principal payment, P_k, and a scheduled interest payment, I_k

– we will use this observation later to create principal-only and interest-only MBS.

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Prepayment Risk and Mortgage Pass-Throughs

M. Haugh G. Iyengar

Department of Industrial Engineering and Operations Research
Columbia University

Prepayment Risk

Many mortgage-holders in the US are allowed to pre-pay the mortgage principal earlier than **scheduled**

– payments made in excess of the scheduled payments are called prepayments.

There are many possible reasons for prepayments:

1. homeowners must prepay entire mortgage when they sell their home
2. homeowners can **refinance** their mortgage at a better interest rate
3. homeowners may default on their mortgage payments
   – if mortgage is insured then insurer will prepay the mortgage
4. home may be destroyed by flooding, fire etc.
   – again insurance proceeds will prepay the mortgage.

Prepayment modeling is therefore an important feature of pricing MBS

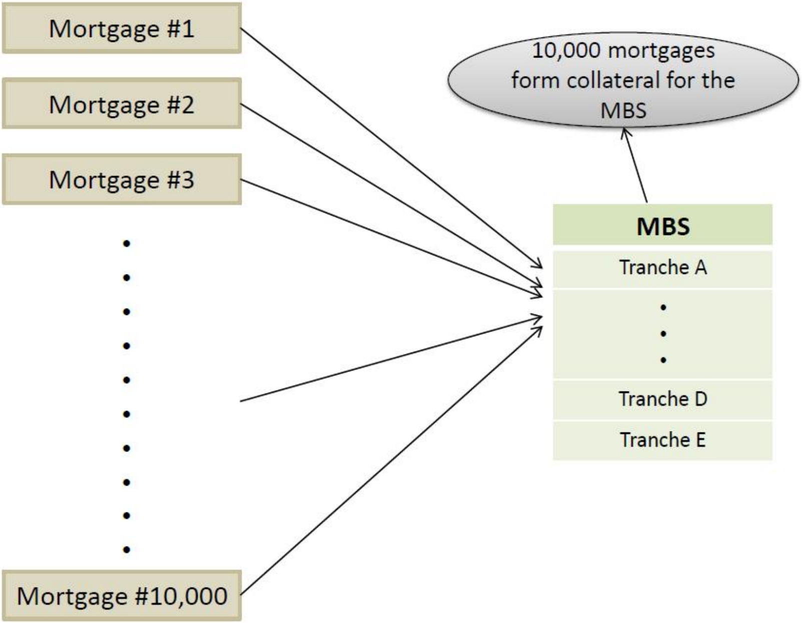
– and the value of some MBS is extremely dependent on prepayment behavior.

Will now consider the simplest type of MBS

– the mortgage pass-through.

Mortgage Pass-Throughs

- In practice, mortgages are often sold on to third parties who can then pool these mortgages together to create mortgage-backed securities (MBS).
- In the US the third parties are either government sponsored agencies (GSAs) such as Ginnie Mae, Freddie Mac or Fannie Mae, or other non-agency third parties such as commercial banks.
- MBS that are issued by the government-sponsored agencies are guaranteed against default
  - not true of non-agency MBS.
- The modeling of MBS therefore depends on whether they are agency or non-agency MBS.
- The simplest type of MBS is the pass-through MBS where a group of mortgages are pooled together.
- Investors in this MBS receive monthly payments representing the interest and principal payments of the underlying mortgages.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Mortgage Pass-Throughs

- The pass-through coupon rate, however, is strictly less than than the average coupon rate of the underlying mortgages
  - due to fees associated with servicing the mortgages.
- Will assume that our MBS are agency-issued and are therefore default-free.

Definition. The weighted average coupon rate (WAC) is a weighted average of the coupon rates in the mortgage pool with weights equal to mortgage amounts still outstanding.

Definition. The weighted average maturity (WAM) is a weighted average of the remaining months to maturity of each mortgage in the mortgage pool with weights equal to the mortgage amounts still outstanding.

Prepayment Conventions

There are important prepayment conventions that are often used by market participants when quoting yields and prices of MBS.

– but first need some definitions.

Definition. The conditional prepayment rate (CPR) is the annual rate at which a given mortgage pool prepays. It is expressed as a percentage of the current outstanding principal level in the underlying mortgage pool.

Definition. The single-month mortality rate (SMM) is the CPR converted to a monthly rate assuming monthly compounding.

The CPR and SMM are therefore related by

$$\begin{aligned} \mathsf{SMM} &= 1 - (1 - \mathsf{CPR})^{1/12} \\ \mathsf{CPR} &= 1 - (1 - \mathsf{SMM})^{12}. \end{aligned}$$

Prepayment Conventions

In practice, the CPR is **stochastic** and depends on the mortgage pool and other economic variables.

However, market participants often use a **deterministic** prepayment schedule as a mechanism to quote MBS yields and so-called option-adjusted spreads etc.

The standard benchmark is the Public Securities Association (PSA) benchmark.

The PSA benchmark assumes the following for 30 year mortgages:

$$\text{CPR} = \begin{cases} 6\% \times (t/30), & \text{if } t \leq 30 \\ 6\%, & \text{if } t > 30. \end{cases}$$

where t is the number of months since the mortgage pool originated.

Slower or faster prepayment rates are then given as some percentage or multiple of PSA.

The Average Life of an MBS

Given a particular prepayment assumption the average life of an MBS is defined as

$$\text{Average Life} \;=\; \sum_{k=1}^{T} \frac{kP_k}{12 \times TP} \tag{6}$$

- where P_k is the principal (scheduled and projected prepayment) paid at time k
- TP is the total principal amount
- T is the total number of months
- and we divide by 12 so that average life is measured in years.

It is immediate that the average life decreases as the PSA speed increases.

Mortgage Yields

- In practice the price of a given MBS security is observed in the market place and from this a corresponding yield-to-maturity can be determined.
- This yield is the interest rate that will make the present value of the expected cash-flows equal to the market price.
- The expected cash-flows are determined based on some underlying prepayment assumption such as 100 PSA, 300 PSA etc.
  - so any quoted yield must be with respect to some prepayment assumptions.
- When the yield is quoted as an annual rate based on semi-annual compounding it is called a bond-equivalent yield.
- Yields are clearly very limited when it comes to evaluating an MBS and indeed fixed-income securities in general.
- Indeed the option-adjusted-spread (OAS) is the market standard for quoting yields on MBS and indeed other fixed income securities with embedded options.

Prepayment Risks for Mortgage Pass-Throughs

- An investor in an MBS pass-through is of course exposed to interest rate risk in that the present value of any **fixed** set of cash-flows decreases as interest rates increase.
- However, a pass-through investor is also exposed to prepayment risk, in particular contraction risk and extension risk.

When interest rates decline prepayments tend to increase and the additional prepaid principal can only be invested at lower interest rates

– this is contraction risk.

When interest rates increase, prepayments tend to decrease and so there is less prepaid principal that can be invested at the higher rates

– this is extension risk.

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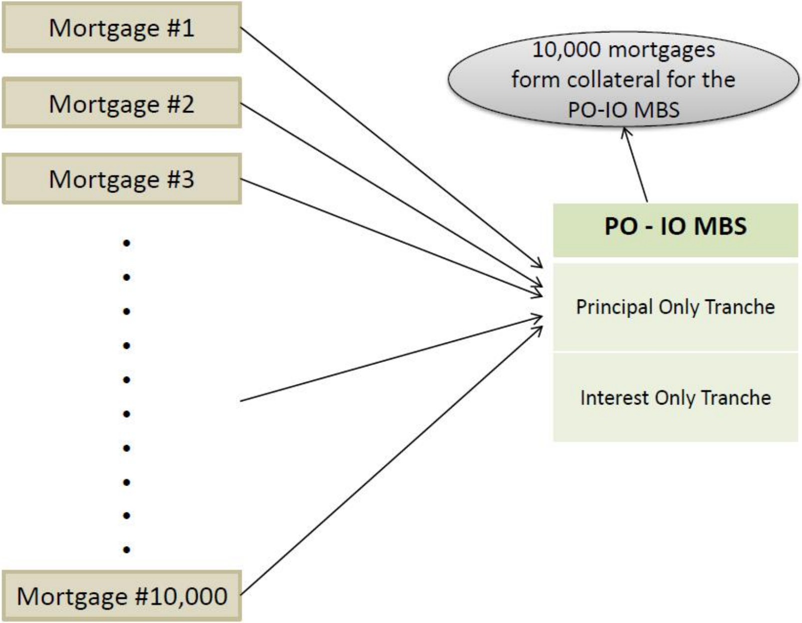
Principal-Only and Interest-Only MBS

M. Haugh **G. Iyengar**

Department of Industrial Engineering and Operations Research
Columbia University


Mortgage #1
Mortgage #2
Mortgage #3
Mortgage #10,000
10,000 mortgages form collateral for the PO-IO MBS
PO - IO MBS
Principal Only Tranche
Interest Only Tranche

Principal-Only and Interest-Only MBS

Since we know M_{k-1} we can compute the interest

$$I_k \;:=\; cM_{k-1}$$

on M_{k-1} that would be due in the next period, i.e. period k.

This also means we can interpret the k^{th} payment as paying

$$P_k :=\; B - cM_{k-1}$$

of the remaining principal, M_{k-1}.

Now recall our earlier expression for M_k:

$$M_k \;=\; (1+c)^k M_0 - B\left[\frac{(1+c)^k - 1}{c}\right]. \tag{7}$$

Using (7), we therefore obtain

$$\begin{aligned} P_k \;&=\; B - c\left((1+c)^{k-1}M_0 - B\left[\frac{(1+c)^{k-1}-1}{c}\right]\right) \\ &=\; (B - c\,M_0)\,(1+c)^{k-1}. \end{aligned}$$

Principal-Only MBS

The present value, V_0, of the **principal payment stream** is therefore given by

$$V_0 \;=\; (B - cM_0)\,\frac{(1+r)^n - (1+c)^n}{(r-c)(1+r)^n} \tag{8}$$

where, as before, r, is the per-period risk-free interest rate.

Can use l'Hôpital's rule to check that

$$\lim_{c \to r} V_0 \;=\; \frac{n(B - rM_0)}{1+r}.$$

Now recall our earlier expression:

$$B = \frac{c(1+c)^n M_0}{(1+c)^n - 1}. \tag{9}$$

If we use (9) to substitute for B in (8) then we obtain

$$V_0 \;=\; \frac{cM_0}{(1+c)^n - 1} \times \frac{(1+r)^n - (1+c)^n}{(r-c)(1+r)^n}. \tag{10}$$

Principal-Only MBS

In the case $r = c$ (10) reduces to

$$V_0 \;=\; \frac{rn\,M_0}{(1+r)\left[(1+r)^n - 1\right]}. \tag{11}$$

It is clear that the earlier mortgage payments comprise of interest payments rather than principal payments

– only later in the mortgage is this relationship reversed.

Indeed this property is reflected in the fact that

$$\lim_{n\to\infty} V_0 = 0.$$

Interest-Only MBS

We can also compute the present value, W_0 say, of the **interest payment stream**

– again assuming there are no mortgage prepayments.

To do this we could compute the sum

$$W_0 \;=\; \sum_{k=1}^{n} \frac{I_k}{(1+r)^k}$$

– but much easier to recognize that the sum of the principal-only and interest-only streams must equal the total value of the mortgage, F_0.

Now recall our earlier expression for F_0:

$$F_0 \quad = \quad \frac{c(1+c)^n M_0}{(1+c)^n - 1} \times \frac{(1+r)^n - 1}{r(1+r)^n}. \tag{12}$$

Interest-Only MBS

Since $W_0 = F_0 - V_0$ we can use (12) and (10) to obtain

$$W_0 \;=\; \frac{cM_0}{[(1+c)^n - 1](1+r)^n} \left[(1+c)^n \, \frac{(1+r)^n - 1}{r} \;-\; \frac{(1+r)^n - (1+c)^n}{r-c}\right].$$

Moreover when $r \rightarrow c$ it is easy to check that this reduces to

$$W_0 \;=\; M_0 \;-\; \frac{rn\, M_0}{(1+r)\,[(1+r)^n - 1]}$$

as expected from (11) and since $F_0 = M_0$ when $r = c$.

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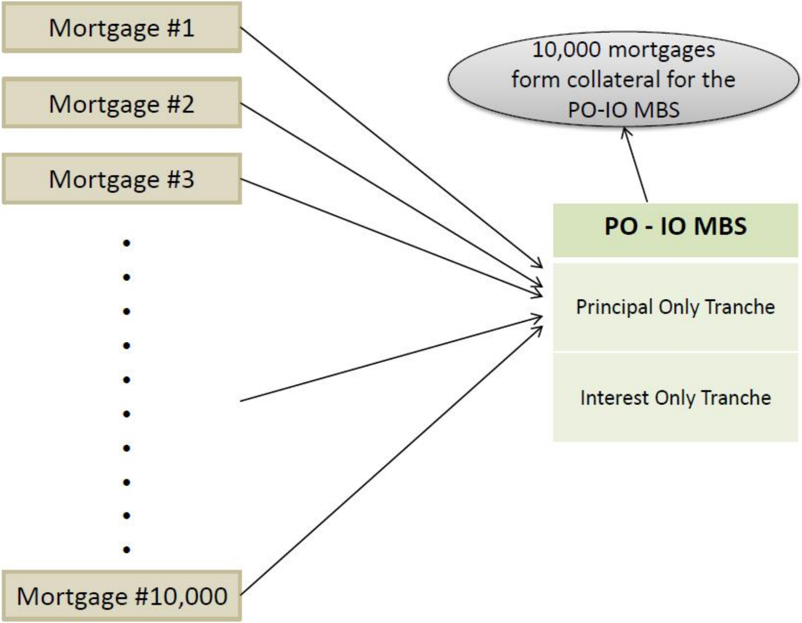
Risks of Principal-Only and Interest-Only MBS

M. Haugh **G. Iyengar**

Department of Industrial Engineering and Operations Research
Columbia University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PO - IO MBS
Principal Only Tranche
Interest Only Tranche

Duration of the Principal-Only MBS

Again we assume no prepayments and consider the **durations** of the PO and IO cash-flow streams.

The duration of a cash-flow is a weighted average of the times at which each component of the cash-flow is received

– a standard measure of the **risk** of a cash-flow.

It should be clear that the principal stream has a longer duration than the interest stream.

If we let D_P denote the duration of the principal stream, then it is given by

$$D_P = \frac{1}{12\, V_0} \sum_{k=1}^{n} \frac{k\, P_k}{(1+r)^k} \tag{13}$$

where we divide by 12 in (13) to convert the duration into annual rather than monthly time units.

Duration of the Interest-Only MBS

Similarly, we can compute the duration, D_I, of the interest-only stream as

$$\begin{aligned} D_I &= \frac{1}{12\,W_0} \sum_{k=1}^{n} \frac{k\,I_k}{(1+r)^k} \\ &= \frac{1}{12\,W_0} \sum_{k=1}^{n} \frac{k\,(B-P_k)}{(1+r)^k} \\ &= \frac{1}{12\,W_0} \sum_{k=1}^{n} \frac{k\,B}{(1+r)^k} - \frac{V_0}{W_0}\,D_P. \end{aligned} \tag{14}$$

Principal-Only and Interest-Only MBS in Practice

To this point we have assumed that prepayments do not occur.

But this is not realistic: in practice pass-throughs do experience prepayments and the PO and IO cash-flows must reflect these prepayments correctly.

But this is straightforward: the interest payment in period k is simply, as before,

$$I_k := cM_{k-1}$$

where M_{k-1} is the mortgage balance at the end of period $k-1$.

M_k must now be calculated iteratively on a path-by-path basis as

$$M_k \;=\; M_{k-1} \;-\; \text{ScheduledPrincipalPayment}_k \;-\; \text{Prepayment}_k,$$

for $k = 1, \ldots, n$ and where $\text{ScheduledPrincipalPayment}_k$ is now the scheduled principal payment (adjusted for earlier prepayments) in period k.

The Risks of PO and IO MBS

The **risk profiles** of principal-only and interest-only securities are very different from one another.

The principal-only investor would like prepayments to **increase**.

The interest-only investor wants prepayments to **decrease**

- after all the IO investor only earns interest at time k on the mortgage balance remaining at time k.

In fact the IO security is that rare fixed income security whose price tends to follow the general level of interest rates:

- when interest rates fall the value of the IO security tends to decrease
- and when interest rates increase the expected cash-flow increases due to fewer prepayments but the discount factor decreases
  - the **net effect** can be a rise or fall in value of the IO security.

Question: What happens to the value of a PO security when interest rates (i) increase and (ii) decrease?

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Collateralized Mortgage Obligations (CMOs)

M. Haugh **G. Iyengar**

Department of Industrial Engineering and Operations Research
Columbia University

Collateralized Mortgage Obligations (CMOs)

Collateralized mortgage obligations (CMOS) are mortgage-backed securities that have been created by redirecting the cash-flows from other mortgage securities

– created mainly to mitigate prepayment risk and create securities that are better suited to potential investors.

In practice CMOs are often created from pass-through's but they can also be created from other MBS including, for example, principal-only MBS.

There are many types of CMOs including

- sequential CMOs
- CMOs with accrual bonds
- CMOs with floating-rate and inverse-floating-rate tranches
- planned amortization class (PAC) CMOs.

We'll briefly describe sequential CMOs.

Sequential CMOs

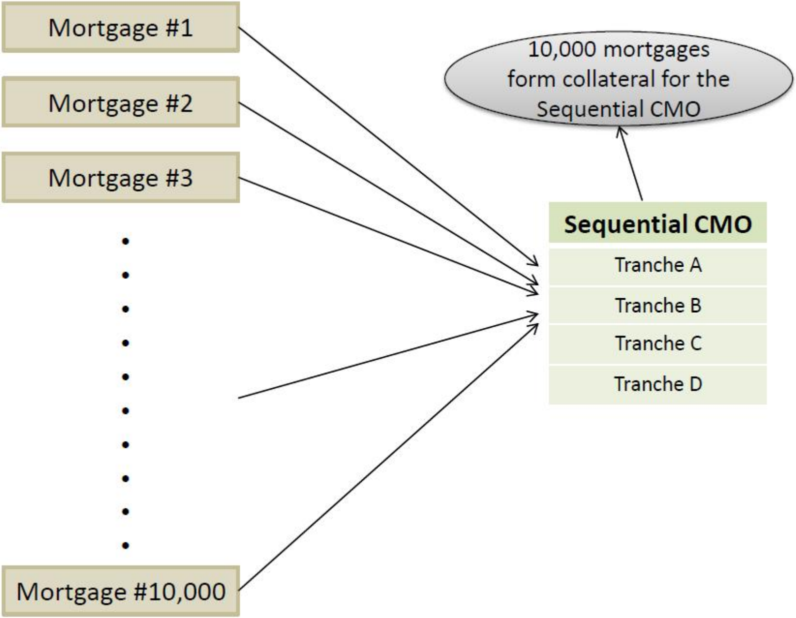
The basic structure of a sequential CMO is that there are several tranches which are ordered in such a way that they are **retired sequentially**.

For example, the payment structure of a sequential CMO with tranches A, B, C and D might be as follows:

Sequential CMO Payment Structure

1. Periodic coupon interest is disbursed to each tranche on the basis of the amount of principal outstanding in the tranche at the beginning of the period.

2. All principal payments are disbursed to tranche A until it is paid off entirely. After tranche A has been paid off all principal payments are disbursed to tranche B until it is paid off entirely. After tranche B has been paid off all principal payments are disbursed to tranche C until it is paid off entirely. After tranche C has been paid off all principal payments are disbursed to tranche D until it is paid off entirely.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Mortgage #2
Mortgage #3
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10,000 mortgages form collateral for the Sequential CMO
Sequential CMO
Tranche A
Tranche B
Tranche C
Tranche D

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Pricing Mortgage-Backed-Securities

M. Haugh G. Iyengar

Department of Industrial Engineering and Operations Research
Columbia University

Prepayment Modeling

- In many respects, the **prepayment model** is the most important feature of any residential MBS pricing engine.
- Term-structure models are well understood in the financial engineering community
  - but this is not true of prepayment models.
- The main problem is that there is relatively little publicly available information concerning prepayments rates
  - so very difficult to **calibrate** prepayment models.
- One well known **publicly available** model is due to Richard and Roll (1989)
  - they model the conditional prepayment rate (CPR) whose definition we recall:

Definition. The CPR is the rate at which a given mortgage pool prepays. It is expressed as a percentage of the **current** outstanding principal level in the underlying mortgage pool.

A Particular Prepayment Model (Richard and Roll 1989)

The model of Richard and Roll assumes

$$CPR_k \;=\; RI_k \times AGE_k \times MM_k \times BM_k$$

where

- RI_k is the refinancing incentive

  $$\textbf{e.g.} \quad RI_k \;=\; .28 \;+\; .14\;\tan^{-1}\left(-8.57 \;+\; 430\left(WAC - r_k(10)\right)\right) \qquad (15)$$

  where $r_k(10)$ is the prevailing 10-year spot rate at time k.

- AGE_k is the seasoning multiplier, **e.g.** $AGE_k = \min\left(1,\; t/30\right)$

- MM_k is the monthly multiplier.

- BM_k is the burnout multiplier

  $$\textbf{e.g.} \quad BM_k \;=\; .3 + .7\,\frac{M_{k-1}}{M_0}$$

  where M_k is the remaining principal balance at time k.

Choosing a Term Structure Model

- We also need to specify a term-structure model in order to fully specify the mortgage pricing model.
- The term structure model will be used to:
  (i) discount all of the MBS cash-flows in the usual risk-neutral pricing framework
  (ii) to compute the refinancing incentive according to (15), for example.
- Whatever term-structure model is used, it is important that we are able to compute the relevant interest rates **analytically**
  - for example, $r_{10}(k)$ in the prepayment model of Richard and Roll.
- Such a model would first need to be calibrated to the term structure of interest rates in the market place as well as liquid interest rate derivatives.
- The actual pricing of MBS then requires **Monte-Carlo simulation**
  - very computationally intensive
  - analytic prices not available.

The Financial Crisis

The so-called sub-prime mortgage market played an important role in the financial crisis of 2008-2009.

Sub-prime mortgages are mortgages that are issued to home-owners with very weak credit

- the true credit quality of the home-owners was often hidden
- the mortgages were often ARMs with so-called **teaser rates**
  - very low initial mortgage rates intended to "tease" the home-owner into accepting the mortgage.

The financial engineering aspect of the MBS-ABS market certainly played a role in the crisis

- particularly when combined with the alphabet soup of CDO's and ABS-CDO's
- these products are too complex and too difficult to model.

But there were many other causes including: moral hazard problem of mortgage brokers, ratings agencies, bankers etc, inadequate regulation, inadequate risk management and poor corporate governance.