hw3

Coding the Matrix, Summer 2013

Please fill out the stencil file named "`hw3.py`". While we encourage you to complete the Ungraded Problems, they do not require any entry into your stencil file.

Matrix-Vector Multiplication

Problem 1: Compute the following matrix-vector products (I recommend you not use the computer to compute these):

1. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * [0.5, 0.5]$
2. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} * [1.2, 4.44]$
3. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} * [1, 2, 3]$

Problem 2: What 2×2 matrix M satisfies $M * [x, y] = [y, x]$ for all vectors $[x, y]$?

Problem 3: What 3×3 matrix M satisfies $M * [x, y, z] = [z + x, y, x]$ for all vectors $[x, y, z]$?

Problem 4: What 3-by-3 matrix M satisfies $M * [x, y, z] = [2x, 4y, 3z]$ for all vectors $[x, y, z]$?

Matrix Multiplication: Dimension of Matrices

Problem 5: For each of the following problems, answer whether matrix multiplication is valid or not. If it is valid, give the number of rows and the number of columns of the resulting matrix (you do not need to provide the matrix itself).

1. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$
2. $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix}^T$

4. $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T$

5. $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}^T$

7. $\begin{bmatrix} 2 & 1 & 5 \end{bmatrix}^T \begin{bmatrix} 1 & 6 & 2 \end{bmatrix}$

Matrix-Matrix Multiplication

Problem 6: Compute:

1. $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 6 \\ 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

5. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 1 & -3 \\ 2 & 2 & -2 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ (Remember the superscript T means “transpose”.)

Problem 7: Let

$$A = \begin{bmatrix} 2 & 0 & 1 & 5 \\ 1 & -4 & 6 & 2 \\ 3 & 0 & -4 & 2 \\ 3 & 4 & 0 & -2 \end{bmatrix}$$

For each of the following values of the matrix B, compute AB and BA. (I recommend you not use the computer to compute these.)

1. $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 2. $B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ 3. $B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Problem 8: Let a, b be numbers and let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$.

1. What is AB? Write it in terms of a and b.
2. For a matrix M and a nonnegative integer k, we denote by M^k the k-fold product of M with itself, i.e.

$$\underbrace{MMM \ldots M}_{k \text{ times}}$$

Plug in 1 for a in A. What is A^2, A^3? What is A^n where n is a positive integer?

Problem 9: Let

$$A = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 1 & 5 & -2 & 3 \\ 4 & 4 & 4 & 0 \\ -1 & 6 & 2 & -5 \end{bmatrix}$$

For each of the following values of the matrix B, compute AB and BA without using a computer. (To think about: Which definition of matrix-matrix multiplication is most useful here? What does a nonzero entry at position (i, j) in B contribute to the j^{th} column of AB? What does it contribute to the i^{th} row of BA?)

(a) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Column Vector and Row Vector Matrix Multiplications

Problem 10: Compute the result of the following matrix multiplications.

(a) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 5 & 1 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 & 2 \\ -2 & 6 & 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(e) $\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}^T \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ (Remember the superscript T means "transpose".)

Matrix-vector and vector-matrix multiplication

You will write several procedures, each implementing matrix-vector multiplication using a *specified definition* of matrix-vector multiplication or vector-matrix multiplication.

- These procedures can be written and run after you write `getitem(M, k)` but before you make any other additions to `Mat`.
- These procedures must *not* be designed to exploit sparsity.
- Your code must *not* use the matrix-vector and vector-matrix multiplication operations that are a part of `Mat`.
- Your code should use procedures
  `mat2rowdict`, `mat2coldict`, `rowdict2mat(rowdict)`, and/or `coldict2mat(coldict)`
  from the `matutil` module.

Try reproducing the results below with the procedures you have written:

- $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$
- $\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -5 & 10 \\ -4 & 8 \\ -3 & 6 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -40 & 80 \end{bmatrix}$

Problem 11: Write the procedure `lin_comb_mat_vec_mult(M,v)`, which multiplies `M` times `v` using the linear-combination definition. For this problem, the only operation on `v` you are allowed is getting the value of an entry using brackets: `v[k]`. The vector returned must be computed as a linear combination.

Problem 12: Write `lin_comb_vec_mat_mult(v,M)`, which multiply `v` times `M` using the linear-combination definition. For this problem, the only operation on `v` you are allowed is getting the value of an entry using brackets: `v[k]`. The vector returned must be computed as a linear combination.

Problem 13: Write `dot_product_mat_vec_mult(M,v)`, which multiplies `M` times `v` using the dot-product definition. For this problem, the only operation on `v` you are allowed is taking the dot-product of `v` and another vector and `v`: `u*v` or `v*u`. The entries of the vector returned must be computed using dot-product.

Problem 14: Write `dot_product_vec_mat_mult(v,M)`, which multiplies `v` times `M` using the dot-product definition. For this problem, the only operation on `v` you are allowed is taking the dot-product of `v` and another vector and `v`: `u*v` or `v*u`. The entries of the vector returned must be computed using dot-product.

Matrix-matrix multiplication definitions

You will write several procedures, each implementing matrix-matrix multiplication using a *specified definition* of matrix-matrix multiplication.

- These procedures can be written and run only after you have written and tested the procedures in `mat.py` that perform matrix-vector and vector-matrix multiplication.
- These procedures must *not* be designed to exploit sparsity.
- Your code must *not* use the matrix-matrix multiplication that is a part of `Mat`. For this reason, you can write these procedures before completing that part of `Mat`.
- Your code should use procedures
  `mat2rowdict`, `mat2coldict`, `rowdict2mat(rowdict)`, and/or `coldict2mat(coldict)`
  from the `matutil` module.

Problem 15: `Mv_mat_mat_mult(A,B)`, using the matrix-vector multiplication definition of matrix-matrix multiplication. For this procedure, the only operation you are allowed to do on `A` is matrix-vector multiplication, using the `*` operator: `A*v`. Do *not* use the named procedure `matrix_vector_mul` or any of the procedures defined in the previous problem.

Problem 16: `vM_mat_mat_mult(A,B)`, using the vector-matrix definition. For this procedure, the only operation you are allowed to do on `A` is vector-matrix-vector multiplication, using the `*` operator: `v*A`. Do *not* use the named procedure `vector_matrix_mul` or any of the procedures defined in the previous problem.

Problem 17: `dot_product_mat_mat_mult(A,B)`, using the dot-product definition.
For this procedure, the entries of the matrix returned should be obtained as dot-products of rows/columns of `A` and `B`.

The Inverse of a 2x2 Matrix

Problem 18:

1. Use a formula given in lecture to solve the linear system $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
2. Use the formula to solve the linear system $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3. Use your solutions to find a 2×2 matrix M such that $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ times M is an identity matrix.

4. Calculate M times $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ and calculate $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ times M, and use the following criterion to decide whether M is the inverse of $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$:

Matrices A and B are inverses of each other if and only if both AB and BA are identity matrices.

Matrix Inverse Criterion

Problem 19: For each of the parts below, use

Matrices A and B are inverses of each other if and only if both AB and BA are identity matrices.

to demonstrate that the pair of matrices given are or are not inverse of each other.

1. matrices $\begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -9 & 5 \end{bmatrix}$ over $\mathbb{R}$
2. matrices $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{R}$
3. matrices $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{6} \\ -2 & \frac{1}{2} \end{bmatrix}$ over $\mathbb{R}$
4. matrices $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ over $GF(2)$